Lecture Slides

Essentials of Statistics
5th Edition

and the Triola Statistics Series

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Chapter 4
Probability

Review

Necessity of sound sampling methods.

Common measures of characteristics of data, such as the mean and the standard deviation

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Preview

Rare Event Rule for Inferential Statistics:

If, under a given assumption, the probability of a particular observed event is extremely small, we conclude that the assumption is probably not correct.

Statisticians use the rare event rule for inferential statistics.

Chapter 4
Probability

Key Concept

This section presents three approaches to finding the probability of an event.

The most important objective of this section is to learn how to interpret probability values.

Definitions

- Event

  any collection of results or outcomes of a procedure

- Simple Event

  an outcome or an event that cannot be further broken down into simpler components

- Sample Space

  for a procedure consists of all possible simple events; that is, the sample space consists of all outcomes that cannot be broken down any further

Example

In the following display, we use "b" to denote a baby boy and "g" to denote a baby girl.

Procedure	Example of Event	Sample Space
Single birth	1 girl (simple event)	{b, g}
3 births	2 boys and 1 girl (bbg, bgb, and gbb are all simple events)	{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg}

Notation for Probabilities

P - denotes a probability.

A, B, and C - denote specific events.

$P(A)$ - denotes the probability of event A occurring.

Basic Rules for Computing Probability

Rule 1: Relative Frequency Approximation of Probability

Conduct (or observe) a procedure, and count the number of times event A actually occurs. Based on these actual results, $P(A)$ is approximated as follows:

$$P(A) = \frac{\text{\# of times } A \text{ occurred}}{\text{\# of times procedure was repeated}}$$

Basic Rules for Computing Probability

Rule 2: Classical Approach to Probability (Requires Equally Likely Outcomes)

Assume that a given procedure has n different simple events and that each of those simple events has an equal chance of occurring. If event A can occur in s of these n ways, then

$$P(A) = \frac{s}{n} = \frac{\text{number of ways } A \text{ can occur}}{\text{number of different simple events}}$$

Basic Rules for Computing Probability

Rule 3: Subjective Probabilities

$P(A)$, the probability of event A, is estimated by using knowledge of the relevant circumstances.

Law of Large Numbers

As a procedure is repeated again and again, the relative frequency probability of an event tends to approach the actual probability.

Example

When three children are born, the sample space is: {bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg}

Assuming that boys and girls are equally likely, find the probability of getting three children of all the same gender.

$$P(\text{three children of the same gender}) = \frac{2}{8} = 0.25$$

Simulations

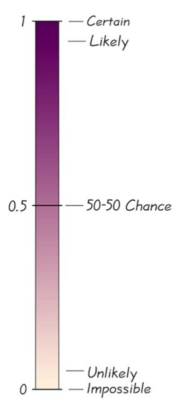
A simulation of a procedure is a process that behaves in the same ways as the procedure itself so that similar results are produced.

Probability Limits

Always express a probability as a fraction or decimal number between 0 and 1.

- The probability of an impossible event is 0.
- The probability of an event that is certain to occur is 1.
- For any event *A*, the probability of *A* is between 0 and 1 inclusive. That is, $0 \le P(A) \le 1$.

Possible Values for Probabilities



Complementary Events

The complement of event *A*, denoted by $\overline{A}$, consists of all outcomes in which the event *A* does not occur.

Example

1010 United States adults were surveyed and 202 of them were smokers.

It follows that:

$$P(\text{smoker}) = \frac{202}{1010} = 0.200$$
$$P(\text{not a smoker}) = 1 - \frac{202}{1010} = 0.800$$

Rounding Off Probabilities

When expressing the value of a probability, either give the exact fraction or decimal or round off final decimal results to three significant digits. (*Suggestion*: When a probability is not a simple fraction such as 2/3 or 5/9, express it as a decimal so that the number can be better understood.) All digits are significant except for the zeros that are included for proper placement of the decimal point.

Definition

An event is **unlikely** if its probability is very small, such as 0.05 or less.

An event has an **usually low number** of outcomes of a particular type or an **unusually high number** of those outcomes if that number is far from what we typically expect.

Odds

The actual odds against event *A* occurring are the ratio $P(\overline{A})/P(A)$, usually expressed in the form of *a*:*b* (or "*a* to *b*"), where *a* and *b* are integers having no common factors.

The actual odds in favor of event *A* occurring are the ratio $P(A)/P(\overline{A})$, which is the reciprocal of the actual odds against the event. If the odds against *A* are *a*:*b*, then the odds in favor of *A* are *b*:*a*.

The payoff odds against event *A* occurring are the ratio of the net profit (if you win) to the amount bet.

payoff odds against event *A* = (net profit) : (amount bet)

Example

If you bet \$5 on the number 13 in roulette, your probability of winning is 1/38 and the payoff odds are given by the casino at 35:1.

a. Find the actual odds against the outcome of 13.

b. How much net profit would you make if you win by betting on 13?

Example - continued

a. Find the actual odds against the outcome of 13.

With *P*(13) = 1/38 and *P*(*not* 13) = 37/38, we get:

$$\text{actual odds against } 13 = \frac{P(\text{not } 13)}{P(13)} = \frac{37/38}{1/38} = \frac{37}{1}$$, or 37:1.

Example - continued

b. Because the payoff odds against 13 are 35:1, we have:

$35 profit for each $1 bet. For a $5 bet, there is $175 net profit. The winning bettor would collect $175 plus the original $5 bet.

Chapter 4
Probability

Key Concept

This section presents the addition rule as a device for finding probabilities that can be expressed as *P*(*A* or *B*), the probability that either event *A* occurs or event *B* occurs (or they both occur) as the single outcome of the procedure.

The key word in this section is "or." It is the *inclusive or*, which means either one or the other or both.

Compound Event

Compound Event

any event combining 2 or more simple events

Notation

P(*A* or *B*) = *P*(in a single trial, event *A* occurs or event *B* occurs or they both occur)

General Rule for a Compound Event

When finding the probability that event *A* occurs or event *B* occurs, find the total number of ways *A* can occur and the number of ways *B* can occur, but find that total in such a way that no outcome is counted more than once.

Compound Event

Formal Addition Rule

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

where *P*(*A* and *B*) denotes the probability that *A* and *B* both occur at the same time as an outcome in a trial of a procedure.

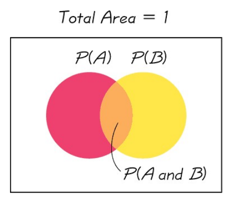
Compound Event

Intuitive Addition Rule

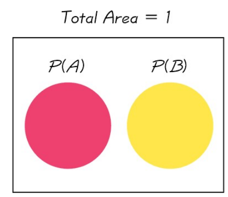
To find $P(A \text{ or } B)$, find the sum of the number of ways event A can occur and the number of ways event B can occur, adding in such a way that every outcome is counted only once. $P(A \text{ or } B)$ is equal to that sum, divided by the total number of outcomes in the sample space.

Disjoint or Mutually Exclusive

Events A and B are disjoint (or mutually exclusive) if they cannot occur at the same time. (That is, disjoint events do not overlap.)



Venn Diagram for Events That Are Not Disjoint



Venn Diagram for Disjoint Events

Complementary Events

A and $\bar{A}$ must be disjoint.

It is impossible for an event and its complement to occur at the same time.

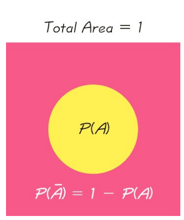
Rule of Complementary Events

$$P(A) + P(\bar{A}) = 1$$

$$P(\bar{A}) = 1 - P(A)$$

$$P(A) = 1 - P(\bar{A})$$

Venn Diagram for the Complement of Event *A*



Chapter 4 Probability

Key Concept

The basic multiplication rule is used for finding $P(A \text{ and } B)$, the probability that event A occurs in a first trial and event B occurs in a second trial.

If the outcome of the first event A somehow affects the probability of the second event B, it is important to adjust the probability of B to reflect the occurrence of event A.

Notation

$P(A \text{ and } B) =$

P(event A occurs in a first trial and event B occurs in a second trial)

$P(B \mid A)$ represents the probability of event B occurring after event A has already occurred.

Formal Multiplication Rule

- $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

Intuitive Multiplication Rule

When finding the probability that event A occurs in one trial and event B occurs in the next trial, multiply the probability of event A by the probability of event B, but be sure that the probability of event B takes into account the previous occurrence of event A.

Caution

When applying the multiplication rule, always consider whether the events are independent or dependent, and adjust the calculations accordingly.

Multiplication Rule for Several Events

In general, the probability of any sequence of independent events is simply the product of their corresponding probabilities.

Dependent and Independent

Two events *A* and *B* are independent if the occurrence of one does not affect the *probability* of the occurrence of the other.

(Several events are similarly independent if the occurrence of any does not affect the probabilities of the occurrence of the others.)

If *A* and *B* are not independent, they are said to be dependent.

Dependent Events

Two events are dependent if the occurrence of one of them affects the *probability* of the occurrence of the other, but this does not necessarily mean that one of the events is a *cause* of the other.

Treating Dependent Events as Independent

Some calculations are cumbersome, but they can be made manageable by using the common practice of treating events as independent when small samples are drawn from large populations. In such cases, it is rare to select the same item twice.

The 5% Guideline for Cumbersome Calculations

If a sample size is no more than 5% of the size of the population, treat the selections as being independent (even if the selections are made without replacement, so they are technically dependent).

Example

Suppose 50 drug test results are given from people who use drugs:

Positive Test Results:	44
Negative Test Results:	6
Total Results:	50

If 2 of the 50 subjects are randomly selected ***without replacement***, find the probability that the first person tested positive and the second person tested negative.

Example – continued

If 2 of the 50 subjects are randomly selected ***without replacement***, find the probability that the first person tested positive and the second person tested negative.

Positive Test Results:	44
Negative Test Results:	6
Total Results:	50

$$P(\text{positive test result for first person}) = \frac{44}{50}$$

$$P(\text{negative test result for second person}) = \frac{6}{49}$$

$$P(\text{1st selection is positive and 2nd is negative}) = \frac{44}{50} \cdot \frac{6}{49} = 0.108$$

Example

When two different people are randomly selected from those in your class, find the indicated probability by assuming birthdays occur on the same day of the week with equal frequencies.

a. Probability that two people are born on the same day of the week.

b. Probability that two people are both born on Monday.

Example – continued

a. Probability that two people are born on the same day of the week.

Because no particular day is specified, the first person can be born on any day. The probability that the second person is born on the same day is 1/7, so the probability both are born on the same day is 1/7.

b. Probability that two people are both born on Monday.

The probability the first person is born on Monday is 1/7, and the same goes for the second person. The probability they are both born on Monday is: $\frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$.

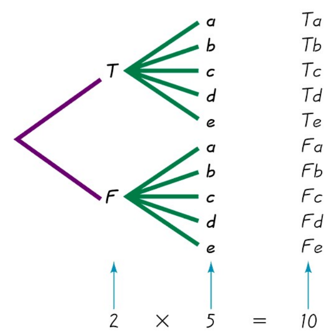
Tree Diagrams

A tree diagram is a picture of the possible outcomes of a procedure, shown as line segments emanating from one starting point. These diagrams are sometimes helpful in determining the number of possible outcomes in a sample space, if the number of possibilities is not too large.

Tree Diagrams

This figure summarizes the possible outcomes for a true/false question followed by a multiple choice question.

Note that there are 10 possible combinations.

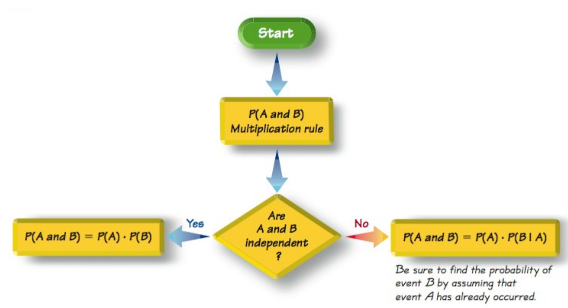


Summary of Fundamentals

- In the addition rule, the word "or" in $P(A \text{ or } B)$ suggests addition. Add $P(A)$ and $P(B)$, being careful to add in such a way that every outcome is counted only once.

- In the multiplication rule, the word "and" in $P(A \text{ and } B)$ suggests multiplication. Multiply $P(A)$ and $P(B)$, but be sure that the probability of event B takes into account the previous occurrence of event A.

Applying the Multiplication Rule



Chapter 4
Probability

Key Concepts

Probability of "at least one":
Find the probability that among several trials, we get at least one of some specified event.

Conditional probability:
Find the probability of an event when we have additional information that some other event has already occurred.

Complements: The Probability of "At Least One"

- "At least one" is equivalent to "one or more."
- The complement of getting at least one item of a particular type is that you get no items of that type.

Finding the Probability of "At Least One"

To find the probability of at least one of something, calculate the probability of none and then subtract that result from 1. That is,

$$P(\text{at least one}) = 1 - P(\text{none}).$$

Example

Topford supplies X-Data DVDs in lots of 50, and they have a reported defect rate of 0.5% so the probability of a disk being defective is 0.005. It follows that the probability of a disk being good is 0.995.

What is the probability of getting at least one defective disk in a lot of 50?

Example – continued

What is the probability of getting at least one defective disk in a lot of 50?

$$P(\text{at least 1 defective disk in 50}) =$$
$$1 - P(\text{all 50 disks are good}) =$$
$$1 - (0.995)^{50} =$$
$$1 - 0.778 = 0.222$$

Conditional Probability

A conditional probability of an event is a probability obtained with the additional information that some other event has already occurred. $P(B \mid A)$ denotes the conditional probability of event *B* occurring, given that event *A* has already occurred, and it can be found by dividing the probability of events *A* and *B* both occurring by the probability of event *A*:

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

Intuitive Approach to Conditional Probability

The conditional probability of *B* given *A* can be found by assuming that event *A* has occurred and then calculating the probability that event *B* will occur.

Example

Refer to the table to find the probability that a subject actually uses drugs, given that he or she had a positive test result.

	Positive Drug Test	Negative Drug Test
Subject Uses Drugs	44 (True Positive)	6 (False Negative)
Subject Does Not Use Drugs	90 (False Positive)	860 (True Negative)

Example - continued

	Positive Drug Test	Negative Drug Test
Subject Uses Drugs	44 (True Positive)	6 (False Negative)
Subject Does Not Use Drugs	90 (False Positive)	860 (True Negative)

$$P(\text{subject uses drugs} \mid \text{subject tests positive}) =$$

$$\frac{P(\text{subject uses drugs and subject tests positive})}{P(\text{subject tests positive})} =$$

$$\frac{44/1000}{134/1000} = \frac{44}{134} = 0.328$$

Confusion of the Inverse

To incorrectly believe that $P(A \mid B)$ and $P(B \mid A)$ are the same, or to incorrectly use one value for the other, is often called confusion of the inverse.

Chapter 4 Probability

Key Concept

In many probability problems, the big obstacle is finding the total number of outcomes, and this section presents several methods for finding such numbers without directly listing and counting the possibilities.

Fundamental Counting Rule

For a sequence of two events in which the first event can occur *m* ways and the second event can occur *n* ways, the events together can occur a total of $m \cdot n$ ways.

Notation

The factorial symbol ! denotes the product of decreasing positive whole numbers.

For example,

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

By special definition, 0! = 1.

Factorial Rule

Number of different permutations (order counts) of *n* different items can be arranged when all *n* of them are selected. (This factorial rule reflects the fact that the first item may be selected in *n* different ways, the second item may be selected in *n – 1* ways, and so on.)

Permutations Rule
(when items are all different)

Requirements:

1. There are *n* different items available. (This rule does not apply if some of the items are identical to others.)
2. We select *r* of the *n* items (without replacement).
3. We consider rearrangements of the same items to be different sequences. (The permutation of ABC is different from CBA and is counted separately.)

If the preceding requirements are satisfied, the number of permutations (or sequences) of *r* items selected from *n* available items (without replacement) is

$$_nP_r = \frac{n!}{(n-r)!}$$

Permutations Rule
(when some items are identical to others)

Requirements:

1. There are *n* items available, and some items are identical to others.
2. We select all of the *n* items (without replacement).
3. We consider rearrangements of distinct items to be different sequences.

If the preceding requirements are satisfied, and if there are n_1 alike, n_2 alike, . . . n_k alike, the number of permutations (or sequences) of all items selected without replacement is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Combinations Rule

Requirements:

1. There are n different items available.
2. We select r of the n items (without replacement).
3. We consider rearrangements of the same items to be the same. (The combination of ABC is the same as CBA.)

If the preceding requirements are satisfied, the number of combinations of r items selected from n different items is

$$_{n}C_{r} = \frac{n!}{(n-r)!r!}$$

Permutations versus Combinations

When different orderings of the same items are to be counted separately, we have a permutation problem, but when different orderings are not to be counted separately, we have a combination problem.

Example

A byte is a sequence of eight numbers, all either 0 or 1.

The number of possible bytes is $2^8 = 256$.

Example

A history pop quiz asks students to arrange the following presidents in chronological order: Hayes, Taft, Polk, Taylor, Grant, Pierce.

If an unprepared student totally guesses, what is the probability of guessing correctly?

$$\text{Possible arrangements: } 6! = 720$$

$$P(\text{guessing correctly}) = \frac{1}{720} = 0.00139$$

Example

In the Pennsylvania Match 6 Lotto, winning the jackpot requires you select six different numbers from 1 to 49. The winning numbers may be drawn in any order. Find the probability of winning if one ticket is purchased.

$$\text{Number of combinations: } {}_{n}C_{r} = \frac{n!}{(n-r)!r!} = \frac{49!}{43!6!} = 13{,}983{,}816$$

$$P(\text{winning}) = \frac{1}{13{,}983{,}816}$$

Chapter 4 Probability

Key Concept

In this section we use simulations as an alternative approach to finding probabilities.

The advantage to using simulations is that we can overcome much of the difficulty encountered when using the formal rules discussed in the preceding sections.

Simulation

A simulation of a procedure is a process that behaves the same way as the procedure, so that similar results are produced.

Simulation Example

Gender Selection In a test of the MicroSort method of gender selection developed by the Genetics & IVF Institute, 127 boys were born among 152 babies born to parents who used the YSORT method for trying to have a baby boy.

In order to properly evaluate these results, we need to know the probability of getting at least 127 boys among 152 births, assuming that boys and girls are equally likely.

Assuming that male and female births are equally likely, describe a simulation that results in the genders of 152 newborn babies.

Solution

One approach is simply to flip a fair coin 152 times, with heads representing females and tails representing males.

Another approach is to use a calculator or computer to randomly generate 152 numbers that are 0s and 1s, with 0 representing a male and 1 representing a female.

The numbers must be generated in such a way that they are equally likely.

Here are typical results:

Simulation Examples

Solution 1:

- Flipping a fair coin 100 times where heads = female
  tails = male

H	H	T	H	T	T	H	H	H	H
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
female	female	male	female	male	male	male	female	female	female

Solution 2:

- Generating 0's and 1's with a computer or calculator where
  0 = male
  1 = female

0	0	1	0	1	1	1	0	0	0
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
male	male	female	male	female	female	female	male	male	male

Random Numbers

In many experiments, random numbers are used in the simulation of naturally occurring events. Below are some ways to generate random numbers:

- A table of random of digits
- STATDISK
- Minitab
- Excel
- TI-83/84 Plus calculator

Random Numbers

STATDISK

Row	1 Ran...
1	7
2	8
3	16
4	38
5	42
6	46
7	68
8	68
9	104
10	117
11	140
12	195
13	204
14	244
15	271
16	274

Minitab

↓	C1	C2
1	38	
2	48	
3	59	
4	71	
5	101	
6	107	
7	122	
8	129	
9	153	
10	153	
11	163	

Random Numbers

Excel

	A
1	15
2	3
3	15
4	362
5	164
6	184
7	158
8	59
9	143
10	85
11	134

TI-83/84 Plus calculator

```
randInt(1,365,25
→L1
{79 206 340 133…
SortA(L1)
              Done
L1
{17 34 46 70 79…
```


Chapter 4
Probability

4-1 Review and Preview

4-2 Basic Concepts of Probability

4-3 Addition Rule

4-4 Multiplication Rule: Basics

4-5 Multiplication Rule: Complements and Conditional Probability

4-6 Counting

4-7 Probabilities Through Simulations

4-8 Bayes' Theorem

Definitions

A prior probability is an initial probability value originally obtained before any additional information is obtained.

A posterior probability is a probability value that has been revised using additional information that is later obtained.

Bayes' Theorem

The probability of event *A*, given that event *B* has subsequently occurred, is

$$P(A \mid B) = \frac{P(A) \cdot P(B \mid A)}{\left[P(A) \cdot P(B \mid A)\right] + \left[P(\bar{A}) \cdot P(B \mid \bar{A})\right]}$$

Example

In Orange County, 51% are males and 49% are females.

One adult is selected at random for a survey involving credit card usage.

a. Find the prior probability that the selected person is male.

b. It is later learned the survey subject was smoking a cigar, and 9.5% of males smoke cigars (only 1.7% of females do). Now find the probability the selected subject is male.

Example – continued

Notation:

$M =$ male $\qquad \bar{M} =$ female

$C =$ cigar smoker $\qquad \bar{C} =$ not a cigar smoker

a. Before the extra information obtained in part (b), we know 51% of the adults are male, so $P(M) = 0.51$.

Example – continued

b. Based on the additional information:

$$P(M) = 0.51$$
$$P(\bar{M}) = 0.49$$
$$P(C \mid M) = 0.095$$
$$P(C \mid \bar{M}) = 0.017$$

We can now apply Bayes' Theorem:

Example – continued

$$P(M \mid C) = \frac{P(M) \cdot P(C \mid M)}{\left[P(M) \cdot P(C \mid M)\right] + \left[P(\bar{M}) \cdot P(C \mid \bar{M})\right]}$$

$$= \frac{0.51 \cdot 0.095}{\left[0.51 \cdot 0.095\right] + \left[0.49 \cdot 0.017\right]}$$

$$= 0.853 \ (\text{rounded})$$

Bayes' Theorem Generalized

The preceding formula used exactly two categories for event *A*, but the formula can be extended to include more than two categories.

We must be sure the multiple events satisfy two important conditions:

1. The events must be disjoint (with no overlapping).
2. The events must be exhaustive, which means they combine to include all possibilities.

